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A THREE-DIMENSIONAL MATHEMATICAL SIMULATOR OF MULTIPHASE
SYSTEMS IN A PETROLEUM RESERVOIR

by

LEONARD FREDERICK KOEDERITZ, 1946-

A DISSERTATION

Presented to the Faculty of the Graduate School of the

UNIVERSITY OF MISSOURI - ROLLA

In Partial Fulfillment of the Requirements for the Degree

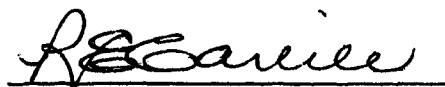
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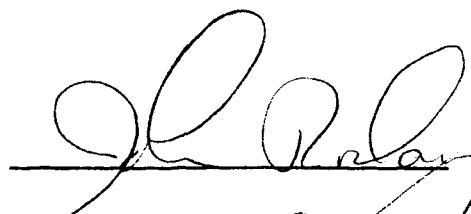
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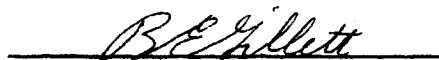


Advisor



M. D. Arnold





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ABSTRACT

A generalized digital analysis for calculating three phase, three dimensional flow in reservoirs is developed. The computerized mathematical model is applicable to depletion drive, gas injection, water injection, and/or aquifer drive systems with modifications specific to the simulation criterion of that type of drive and includes fluid compressibility effects, gravity drainage and capillary pressure. The simulation allows arbitrary consideration of reservoir heterogeneity, geometry, well locations and rates of production. An innovation to the alternating direction technique for linear differential equations discussed by Brian¹ is developed and employed in order to render the method of solution applicable to non-linear differential equations.

The analysis was applied to field-size simulations and was found to be stable for varied drive reservoirs within reasonable time increments.

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I. INTRODUCTION

Digital reservoir simulation has increased in popularity in the last decade, resulting in a voluminous number of one and two dimensional models which vary randomly with the considerations possible; however, progress in three dimensional simulation has been limited, with publications of such being virtually nonexistent. This has been the result of several limiting factors:

- 1) computer size limitations,
- 2) computer time-cost limitations, and
- 3) the fact that most reservoirs can be manipulated such that they can be seemingly well-represented by either one or two dimensional analyses.

The number of grid spaces required for a three dimensional analysis becomes even more formidable than those required for a two dimensional study, especially when an irregular grid network is employed as discussed by van Poolen, et. al.² In addition, the increased grid size requires a greater computational time. Although many petroleum engineers feel that aerial and vertical two dimensional models will simulate both reservoirs and individual wells, these studies do not actually account for flow in three dimensions, and therefore do not fully represent reservoirs containing the following characteristics:

- 1) separated oil zones containing vertical transmissibilities,

- 2) thick oil zones, and/or
- 3) combination drive effects; i.e., oil reservoirs with bottom water drives and/or gas caps overlaying the oil zone.

The purpose of this study is to develop a mathematical simulation model fully capable of accounting for the above conditions and to develop computational techniques which consistently yield stable results for these variable conditions.

II. LITERATURE REVIEW

Model simulation requires that the reservoir be simulated by a grid system wherein each block possesses specific descriptive properties, as explained by Dougherty and Mitchell.³

The primary problem involved in a mathematical simulation model whether it is one or three dimensional is that of solving a large number of simultaneous equations; several techniques have been developed for efficiently solving these large arrays. The most popular to date has been the alternating direction implicit (ADI) technique.⁴ This method has been compared previously to other types of solution for two phase flow and shown to be superior,^{5,6,7} and to yield quite satisfactory results.^{8,9} Three phase flow in two dimensions has also been investigated.¹⁰

Douglas¹¹ has described a three dimensional solution which employs the ADI technique; however, it is time-consuming in that it requires several iterations in order to determine the pressure at a single location. Brian¹ simplified and improved the solution for linear partial differential equations such as transient heat conduction; his technique has been improved so that it is now applicable for the solution of non-linear equations, primarily Darcy's Law.

A word of caution is necessary at this point: a three phase, three dimensional simulator is not the ultimate answer; it is only a necessary extension of two dimensional study for certain cases.

As with any other prediction technique, the results must be correctly interpreted to be of any value.¹²

III. THEORETICAL DEVELOPMENT

Consider a block having dimensions as shown in Figure 1. If this block is in fluid contact on all sides, as in Figure 2, there exist six faces through which flow may occur. Associated with this block are definite rock and fluid properties as well as pressures and saturations which need not be homogeneous with surrounding blocks. This block is one of many which with its unique identifying parameters combine to represent the total reservoir and its flow characteristics.

Based on the principle of conservation of mass, the general flow equation may be expressed as

$$\text{mass in} - \text{mass out} - \text{mass produced} = \text{mass accumulation.} \quad (1)$$

For an individual phase,

$$\text{mass in} = (QpB)_x + (QpB)_y + (QpB)_z,^* \quad (2)$$

where all calculations will be made in pounds per day. The mass leaving a block for any phase may be represented as

$$\begin{aligned} \text{mass out} = \text{mass in} + \Delta x \frac{\partial}{\partial x}(QpB)_x + \Delta y \frac{\partial}{\partial y}(QpB)_y \\ + \Delta z \frac{\partial}{\partial z}(QpB)_z. \end{aligned} \quad (3)$$

Substitution of the reservoir rate, QB , by its equivalent form in Darcy's Law yields the following result, illustrated for the

*All symbols are defined in the Nomenclature (Appendix E).

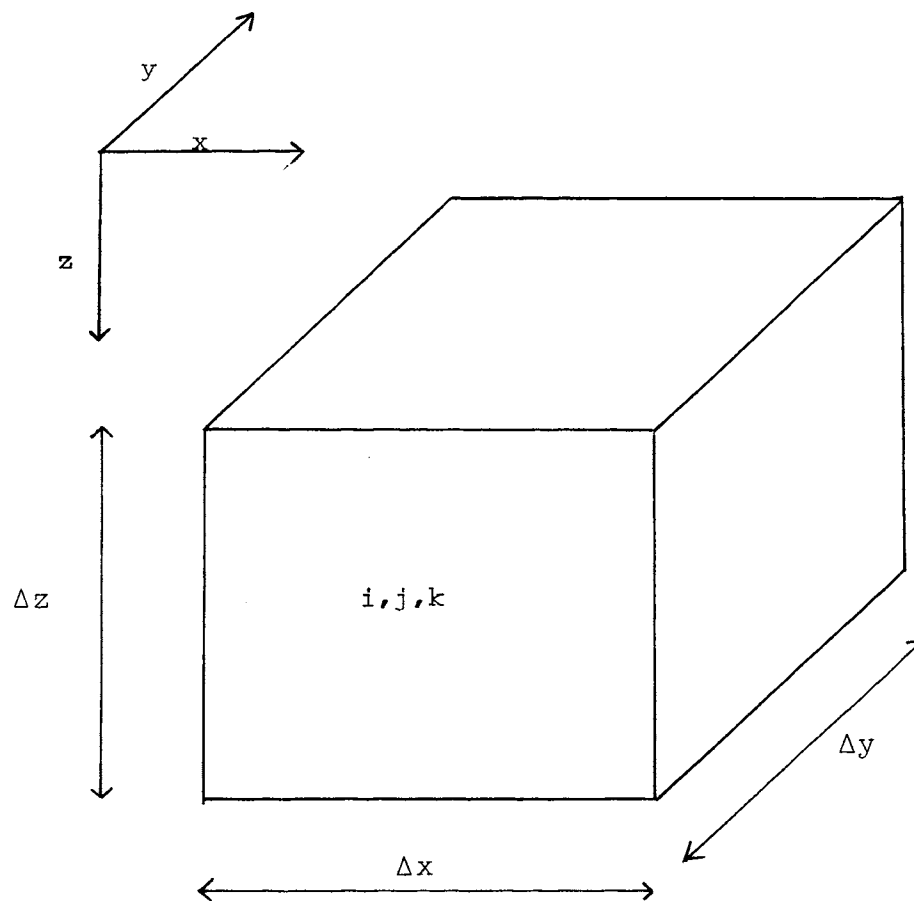


Figure 1 Schematic Representation of an Individual Block

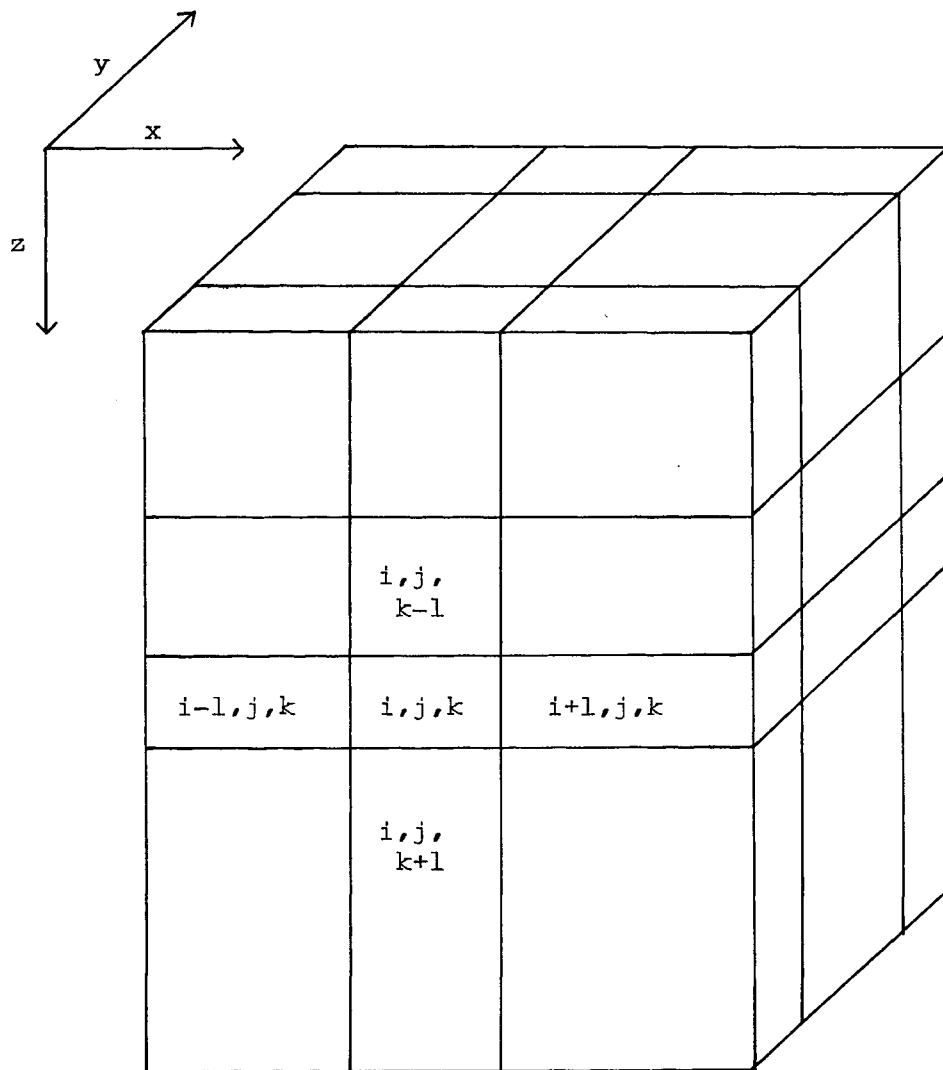


Figure 2 Schematic Representation of the Total Simulator

x-direction only:

$$(QB)_x = -6.3225 \frac{kr}{u} \frac{k}{A_x} \frac{\partial \Phi}{\partial x}. \quad (4)$$

Similar equations exist for the y- and z-directions. The production term for oil may be expressed as

$$\text{oil produced} = 5.615 q_o B_o p_o. \quad (5)$$

Similarly, water injected can be indicated by

$$\text{water injected} = - \text{water produced} = -5.615 q_w B_w p_w. \quad (5a)$$

Substituting Equations (2), (3), (4) and (5) into Equation (1) and rearranging yields the following results, illustrated for the oil phase only:

$$(\text{mass in} - \text{mass out} - \text{mass produced})_o = 6.3225 \Delta x \Delta y \Delta z$$

$$\left[\frac{\partial}{\partial x} \left(\frac{kr_o}{u_o} \frac{k}{p_o} \frac{\partial \Phi_o}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{kr_o}{u_o} \frac{k}{p_o} \frac{\partial \Phi_o}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{kr_o}{u_o} \frac{k}{p_o} \frac{\partial \Phi_o}{\partial z} \right) \right] - 5.615 q_o B_o p_o. \quad (6)$$

A similar equation exists for the water phase. The equation for the gas phase is also the same except that it contains an additional term which accounts for the gas evolved from the oil at pressures below the bubble point. This extra term may be expressed as follows:

$$\text{evolved gas} = - \frac{S_o \phi \Delta x \Delta y \Delta z}{5.615 B_o} B_g p_g \frac{dR_s}{dt}. \quad (7)$$

The total gas term, shown as it would appear on the left-hand side of Equation (1), may be expressed as

$$\begin{aligned}
 & (\text{mass in} - \text{mass out} - \text{mass produced} + \text{mass evolved})_g = \\
 & 6.3225 \Delta x \Delta y \Delta z \left[\frac{\partial}{\partial x} \frac{kr_g k}{u_g} p_g \frac{\partial \Phi_g}{\partial x} \right. \\
 & \left. + \frac{\partial}{\partial y} \frac{kr_g k}{u_g} p_g \frac{\partial \Phi_g}{\partial y} + \frac{\partial}{\partial z} \frac{kr_g k}{u_g} p_g \frac{\partial \Phi_g}{\partial z} \right] \\
 & - Q_g B_g p_g - \Delta x \Delta y \Delta z \phi \frac{S_o B_o p_g}{5.615 B_o} \frac{dR_s}{dt} . \quad (8)
 \end{aligned}$$

The mass accumulation term may be expressed as

$$\begin{aligned}
 \text{mass accumulation} = & \left[S_o \frac{d}{dt} (\Delta x \Delta y \Delta z \phi p_o) + S_w \frac{d}{dt} (\Delta x \Delta y \Delta z \phi p_w) \right. \\
 & \left. + S_g \frac{d}{dt} (\Delta x \Delta y \Delta z \phi p_g) \right] \frac{dP}{dt} . \quad (9)
 \end{aligned}$$

A second equation, completely independent of the mass balance, states that the saturations must sum to unity, i.e.,

$$S_o + S_w + S_g = 1. \quad (10)$$

Taking the time derivative of Equation (10) yields the following result:

$$\frac{d}{dt} (S_o + S_w + S_g) = 0. \quad (10a)$$

Adding the mass balance terms for gas, oil and water, equating them to the accumulation term, and simplifying the result by Equation (10a), yields the following equation describing the time-space function for pressure and phase potentials.

$$\begin{aligned}
6.3225 \Delta x \Delta y \Delta z & \left[\frac{\partial}{\partial x} \left(\frac{kr_o}{u_o} k_{p_o} \right) \frac{\partial \Phi_o}{\partial x} + \frac{\partial}{\partial y} \left(\frac{kr_o}{u_o} k_{p_o} \right) \frac{\partial \Phi_o}{\partial y} \right. \\
& + \frac{\partial}{\partial z} \left(\frac{kr_o}{u_o} k_{p_o} \right) \frac{\partial \Phi_o}{\partial z} + \frac{\partial}{\partial x} \left(\frac{kr_w}{u_w} k_{p_w} \right) \frac{\partial \Phi_w}{\partial x} \\
& + \frac{\partial}{\partial y} \left(\frac{kr_w}{u_w} k_{p_w} \right) \frac{\partial \Phi_w}{\partial y} + \frac{\partial}{\partial z} \left(\frac{kr_w}{u_w} k_{p_w} \right) \frac{\partial \Phi_w}{\partial z} \\
& + \frac{\partial}{\partial x} \left(\frac{kr_g}{u_g} k_{p_g} \right) \frac{\partial \Phi_g}{\partial x} + \frac{\partial}{\partial y} \left(\frac{kr_g}{u_g} k_{p_g} \right) \frac{\partial \Phi_g}{\partial y} \\
& \left. + \frac{\partial}{\partial z} \left(\frac{kr_g}{u_g} k_{p_g} \right) \frac{\partial \Phi_g}{\partial z} \right] - 5.615(q_o B_o p_o + q_w B_w p_w) \\
& - Q_g B_g p_g - \Delta x \Delta y \Delta z \phi \frac{S_o B_g p_g}{5.615 B_o} \frac{dR_s}{dt} = \\
& \left[S_o \frac{d}{dP} \left(\Delta x \Delta y \Delta z \phi p_o \right) + S_w \frac{d}{dP} \left(\Delta x \Delta y \Delta z \phi p_w \right) \right. \\
& \left. + S_g \frac{d}{dP} \left(\Delta x \Delta y \Delta z \phi p_g \right) \right] \frac{dP}{dt} . \quad (11)
\end{aligned}$$

Equation (11) contains 4 independent variables and as such is not solvable. A solution is possible with the following alterations. The potential terms, Φ , may be converted into a combination of pressure and gravity terms as indicated in Appendix A. Employment of the notation of Appendix B for the capillary pressure, gravity, production, compressibility and solution gas terms results in Equation (12) which is now solvable:

$$\begin{aligned}
6.3225 \Delta x \Delta y \Delta z & \left[\frac{\partial}{\partial x} \left(\frac{kr_o}{u_o} k_{p_o} + \frac{kr_w}{u_w} k_{p_w} + \frac{kr_g}{u_g} k_{p_g} \right) \left(\frac{\partial P}{\partial x} \right) \right. \\
& + \frac{\partial}{\partial y} \left(\frac{kr_o}{u_o} k_{p_o} + \frac{kr_w}{u_w} k_{p_w} + \frac{kr_g}{u_g} k_{p_g} \right) \left(\frac{\partial P}{\partial y} \right) \\
& + \frac{\partial}{\partial z} \left(\frac{kr_o}{u_o} k_{p_o} + \frac{kr_w}{u_w} k_{p_w} + \frac{kr_g}{u_g} k_{p_g} \right) \left(\frac{\partial P}{\partial z} \right) \left. \right] \\
& - PCX - PCY - PCZ - QTERM - GZ = G \left(\frac{dP}{dt} \right) . \quad (12)
\end{aligned}$$

The coefficients in Equation (12) are non-linear due to the fact that they are pressure and saturation dependent, and require a special solution of the ADI technique.

Saturation calculations are of the explicit type of solution and are shown in Appendix C.

IV. DISCUSSION OF RESULTS

A set of finite-difference equations was derived and programmed for an IBM 360/50 digital computer. The finite-difference equations describing pressures and saturations as a function of time are shown in Appendices B and C.

A. Stability

Brian,¹ in the development of the theory of employing a tridiagonal matrix technique in the solution of linear differential equations in three dimensions, formulates a possible solution for non-linear equations. His suggestion is to employ either a four-step ADI solution and evaluate the coefficients at the intermediate pressures thusly developed, or to employ a three-step ADI and evaluate all coefficients at the first intermediate pressure. It has been determined that neither of these techniques will yield satisfactory results due to the fact that on the introduction of even minor non-linearity in the flow characteristics, the intermediate pressures become essentially nondescriptive numbers necessary to complete the calculations, but without meaningful relation to actual values. In order to achieve a valid solution, a technique employing an extrapolated pressure was developed and all coefficients were evaluated at the predicted pressure. On completion of the ADI solution, the predicted pressure is compared to the newly calculated pressure and the calculations continue if an allowable tolerance is satisfied, or iteration on the pressures is performed until

convergence occurs. In all cases studied, no iteration was required to simulate the reservoir under consideration. This does not deprive the coefficients of their implicit nature, but attests that the extrapolation technique was adequate.

The inherent requirement for stability in an ADI solution is that the magnitude of the diagonal element, for example, BX in Appendix B, be greater than the magnitude of the sum of the off-diagonal elements, AX and CX , when summing either by rows or columns. For pressures below the bubble point, the inclusion of the gas evolved term in the G -term as defined in Equation B-5b allows the stability requirement to be satisfied even at low gas saturations. When gas is initially being released from solution, the gas compressibility term, a positive number, is not large enough alone to maintain stability in comparison to the oil compressibility term, a negative number. This condition necessitates the inclusion of the gas evolved term in the G -term, which is fundamentally a compressibility function.

The maximum allowable time step required to maintain stability was a five day increment for three-phase systems consisting of 3000 grid points. A ten day time step was found applicable to oil-water fields.

In order to prevent pressure instability early in a study, all production wells must be buffered, or surrounded, by an oil zone. Location of oil-producing wells in this manner eliminates multiple phase flow at the wells prior to the establishment of stable flow conditions.

B. Fluid Flow

The relative rates of flow of the phases are affected by relative and directional absolute permeabilities. Various ratios of vertical to horizontal absolute permeabilities were used with values ranging from 0.01 to 0.50 in this study. The results are shown in Figures 3 and 4.

Figure 3 shows the total field pressure as a function of time in a reservoir with a bottom-water drive having a small aquifer as the source. The field initially contained no free gas and oil was produced at a constant rate over the period indicated, with the study being terminated shortly after water breakthrough. As vertical permeability was decreased, the pressure was better maintained and the cumulative oil produced before water breakthrough was greater. The better pressure maintenance was attributed to the more uniform movement of the water front; consequently, the straight-line relationship should be valid for constant withdrawal rates.

Figure 4 shows the total field pressure as a function of time in a reservoir containing both a weak bottom-water drive and a small gas cap. The oil was produced from a single well at a constant rate. This combination drive field study yielded trends identical to the case previously discussed, although the rate of pressure decline was less. The better sustained pressure was due to the presence of the small gas cap. The analyses of the trends are supported by field observations.¹⁴

Relative permeabilities based on block-averaged saturations were employed for the gas phase; however, block-averaging of saturations

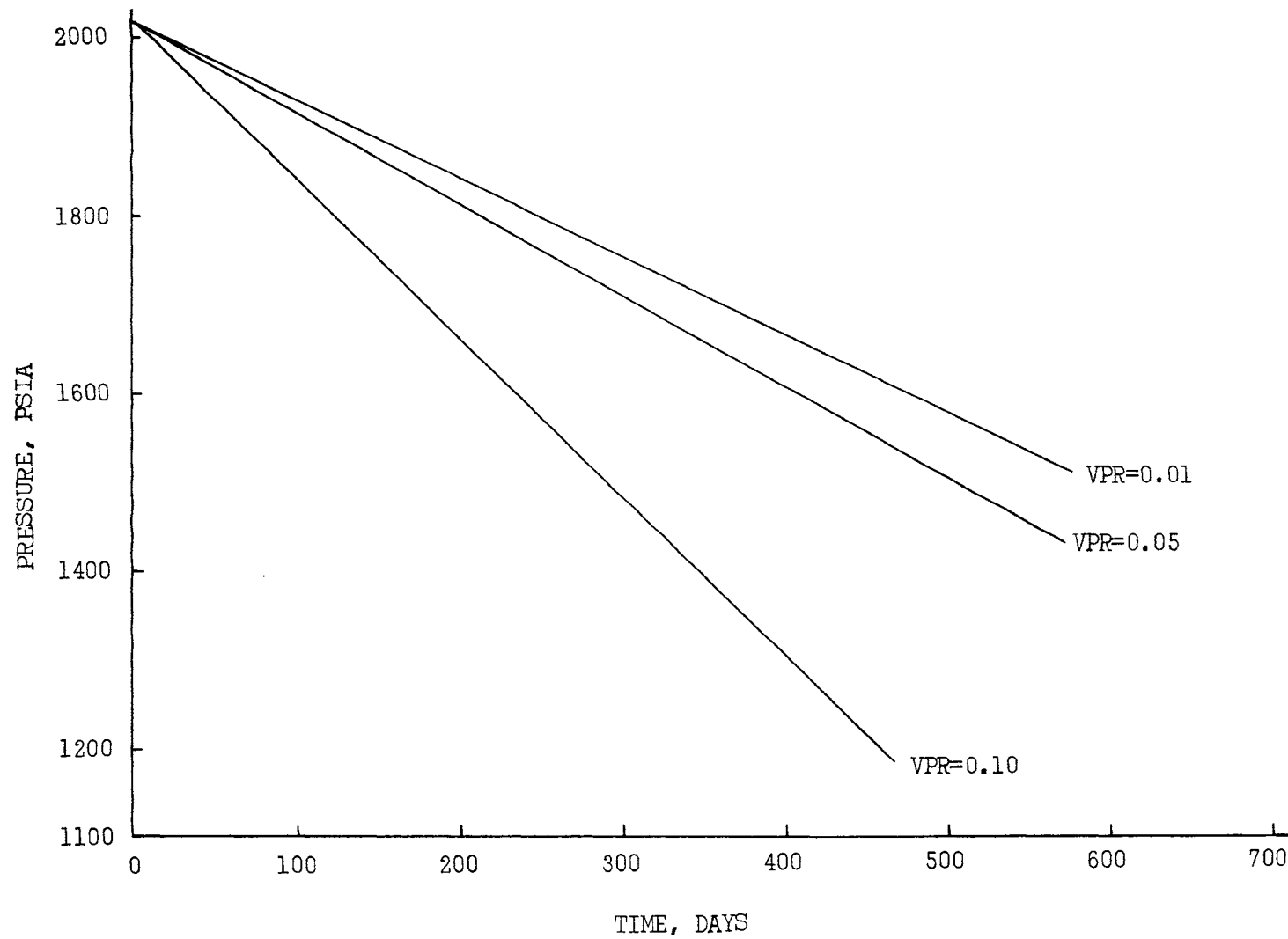


Figure 3 Reservoir Pressure as a Function of Time for a Bottom-Water Drive System

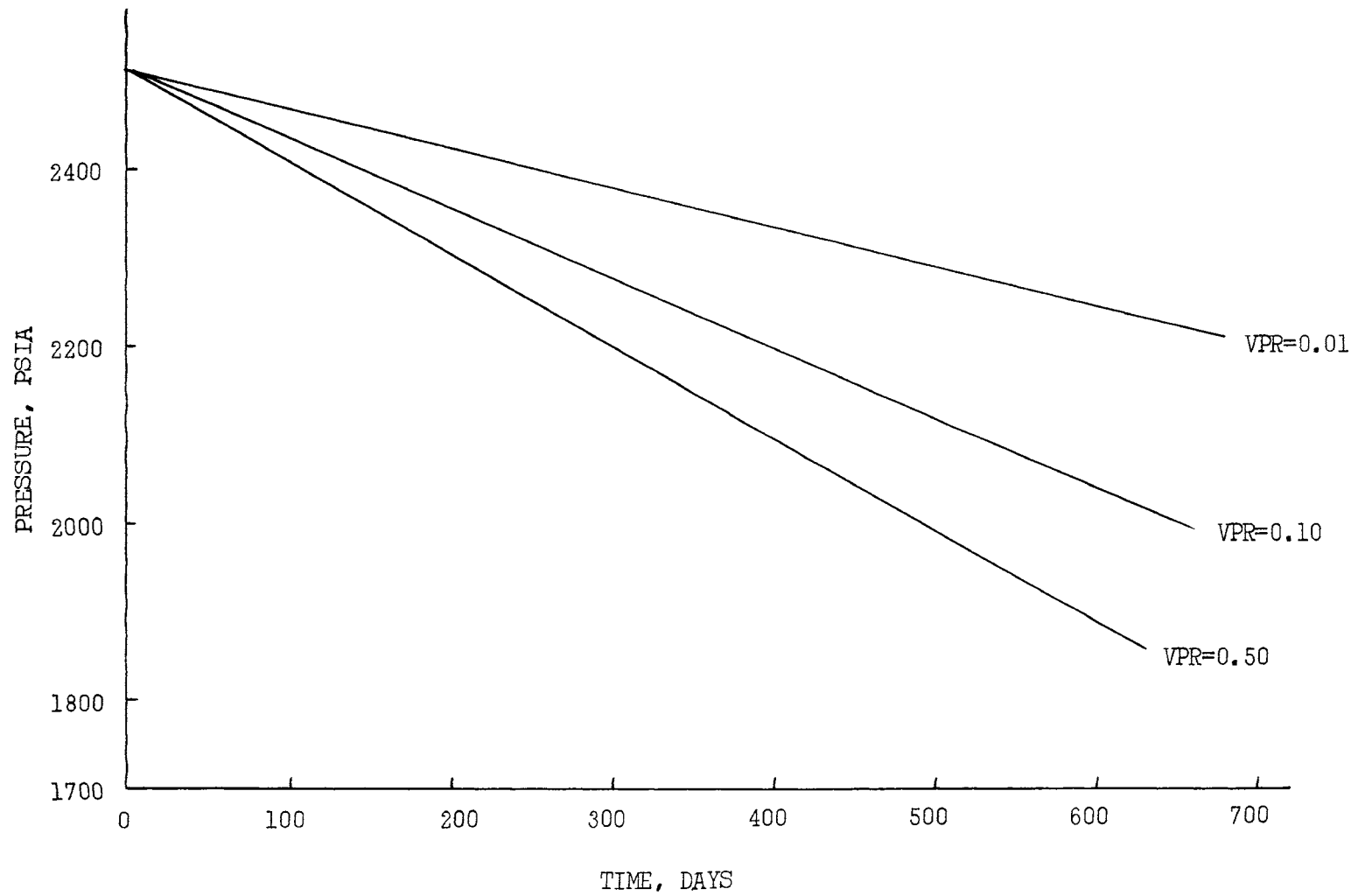


Figure 4 Reservoir Pressure as a Function of Time for a Combination Drive System

for the oil and water phases seriously retarded the flow of these phases. Saturations based on upstream flow locations, as determined by potential changes between blocks, yielded coefficients which realistically described oil and water movement in the reservoir. Employment of upstream-based gas relative permeabilities resulted in serious and unrealistic gas percolation problems.

C. Initial and Boundary Conditions

Initial water saturations were determined at various depths from capillary pressure data; oil and gas saturations were then determined depending on location with respect to the gas-oil contact. The initial pressure distribution was calculated based on gravity and capillarity differences between a point and a measured pressure at a known location; i.e., one pressure in a field and its location must be known in order to generate a pressure distribution.

All outer or exposed surfaces had no-flow boundaries, and as such, the reservoir and any exterior conditions which apply to it, such as an aquifer, must be enclosed within the model. In order that the model conform to the actual reservoir, absolute permeability and/or saturation distributions may be adjusted to create saturation fronts and limited flow conditions.

D. Saturations

A graphical representation in the x-z plane of the water and gas fronts at various time levels for a combination drive system is shown in Figure 5. In this case, the driving mechanism consisted of bottom-water, expanding gas cap, and solution gas. The realistic

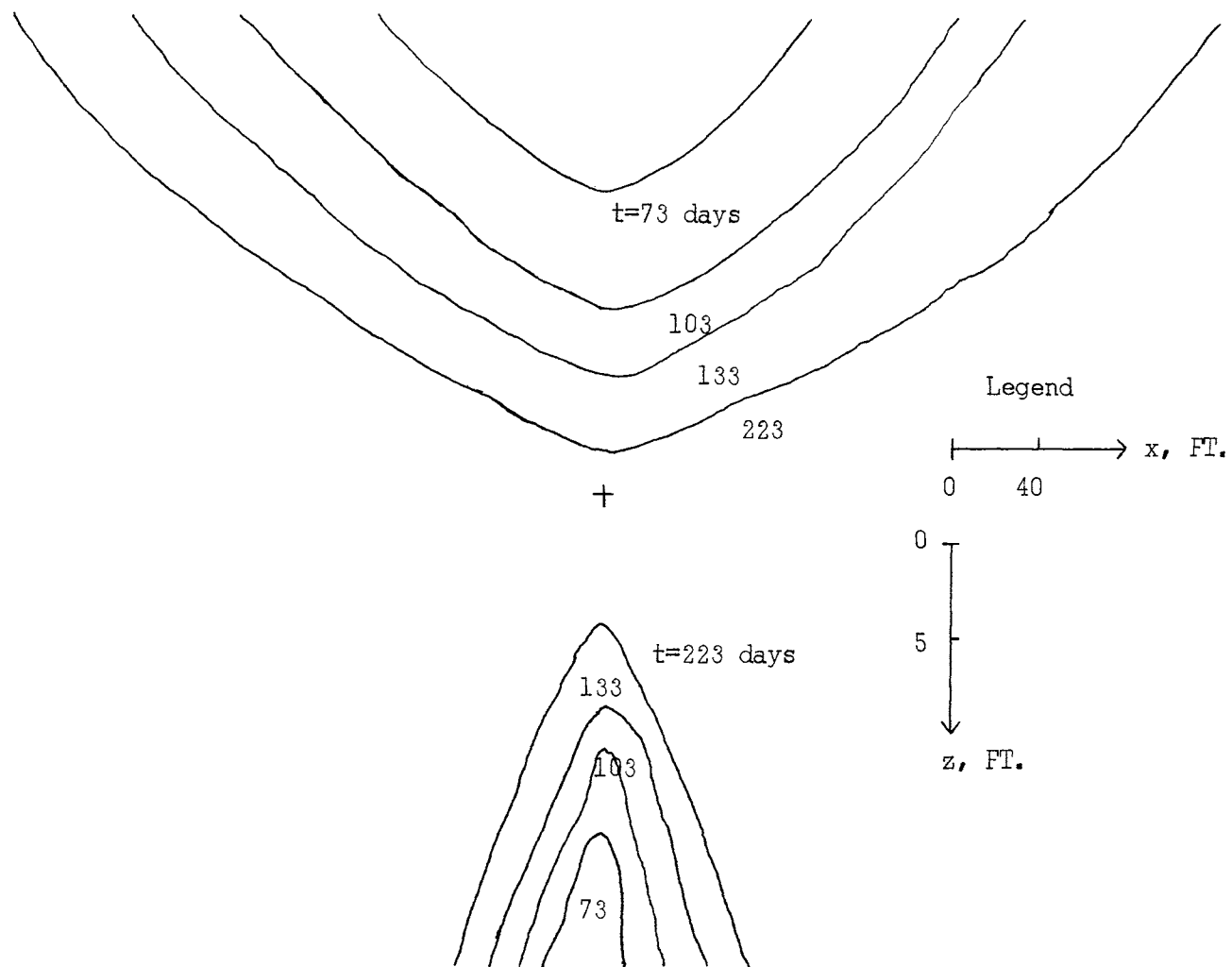


Figure 5 Saturation Fronts in the x - z Plane at Various Times, Combination Drive

formation of the gas and water fronts, respectively, tends to confirm the validity of the simulation technique.

Figure 6 shows the water saturation distribution as a function of time and position. The disturbance shown is caused by a partially penetrating well which reaches total depth at some point above the advancing saturation front. The cone-like formations of the saturation fronts agree with previously established observations.¹⁵

E. Pressures

The pressure distribution around an individual producing well having a bottom-water drive is shown in Figures 7, 8 and 9. The pressure fronts in the aerial view, or x-y plane, as shown in Figures 7 and 8 at varied elapsed times, indicate a radial flow pattern. Figure 9 is a vertical view of the producing well and surrounding areas, in this case, the x-z plane. The elliptic pressure distribution is explained when the limiting vertical permeability factor, 0.01 for this study, and the ratio of horizontal to vertical thicknesses, which ranged from a value of 10 to 1 to values of 30 to 1, are considered.

A pressure distribution for two wells, both in the same horizontal plane and producing at identical rates, is shown in Figure 10. The interference between the wells is indicated by the elongation parallel to the y-axis. This phenomenon is very similar to that reported by Muskat.¹⁶ Also studied were the relationships between injection-production wells and between producing wells at various depths.

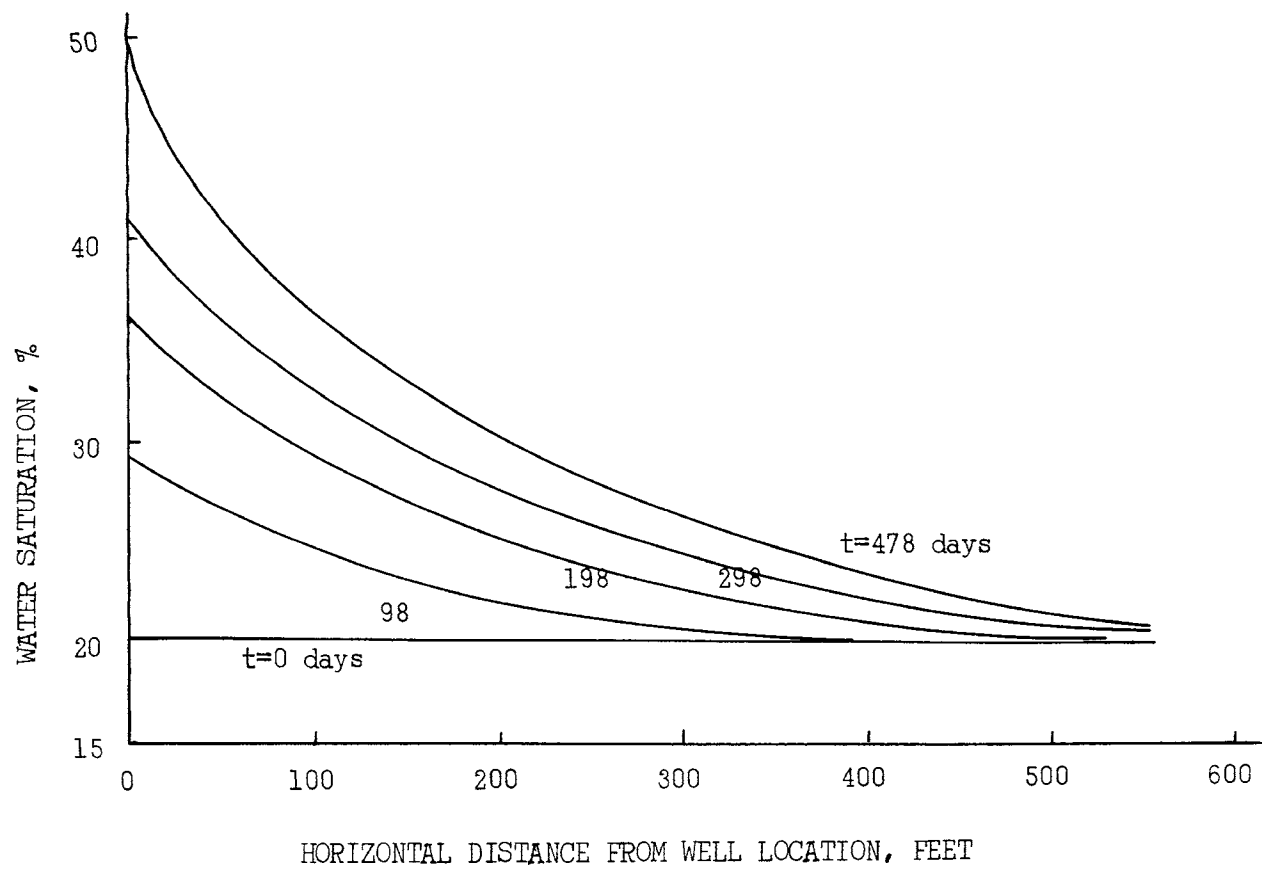


Figure 6 Water Saturation Distribution as a Function of Time and Position

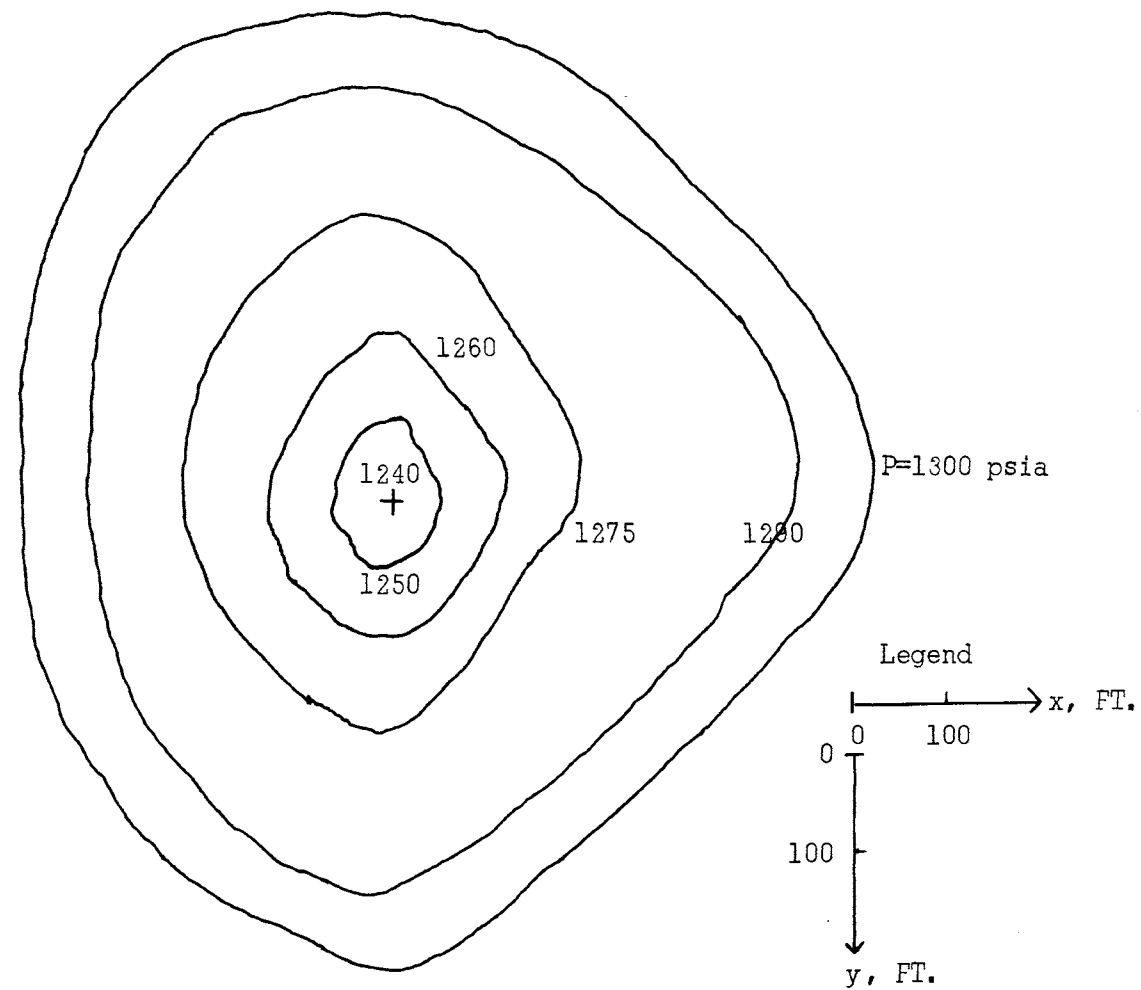


Figure 7 Pressure Distribution Surrounding a Producing Well in a
Bottom-Water Drive Reservoir After 833 Days, x-y Plane

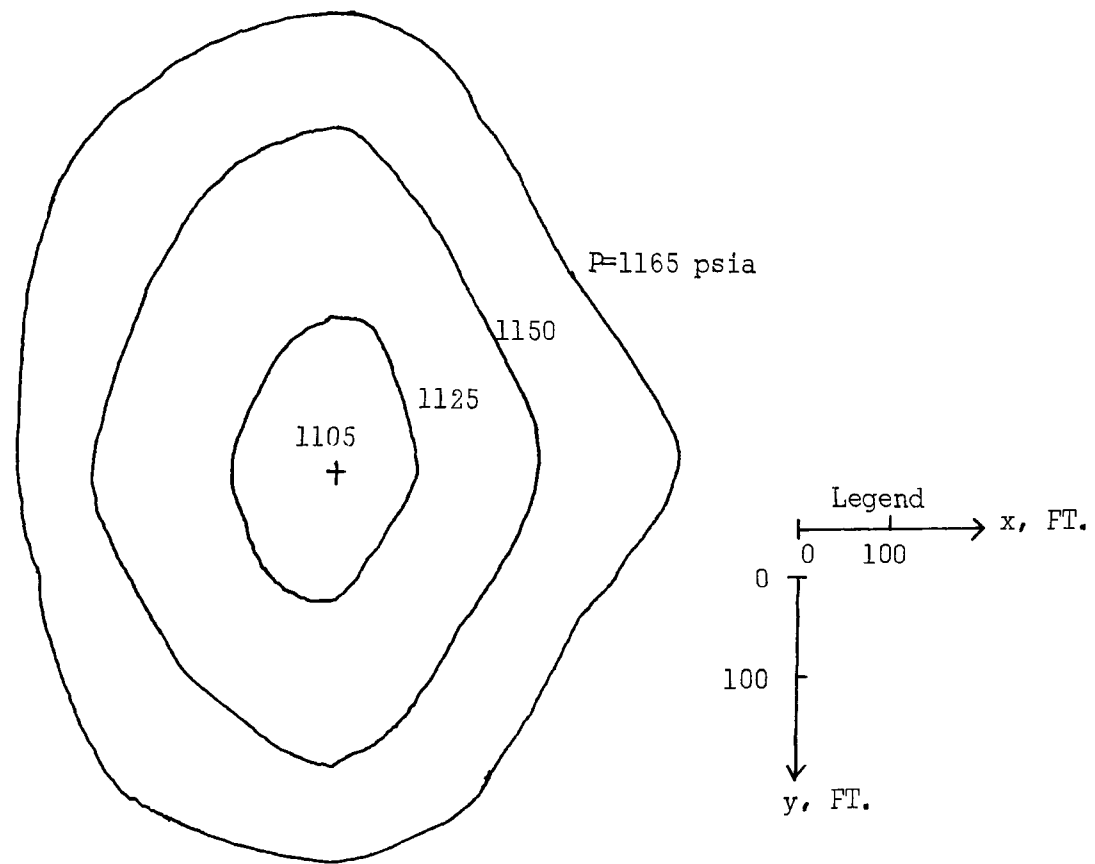


Figure 8 Pressure Distribution Surrounding a Producing Well in a
Bottom-Water Drive Reservoir After 1188 Days, x-y Plane

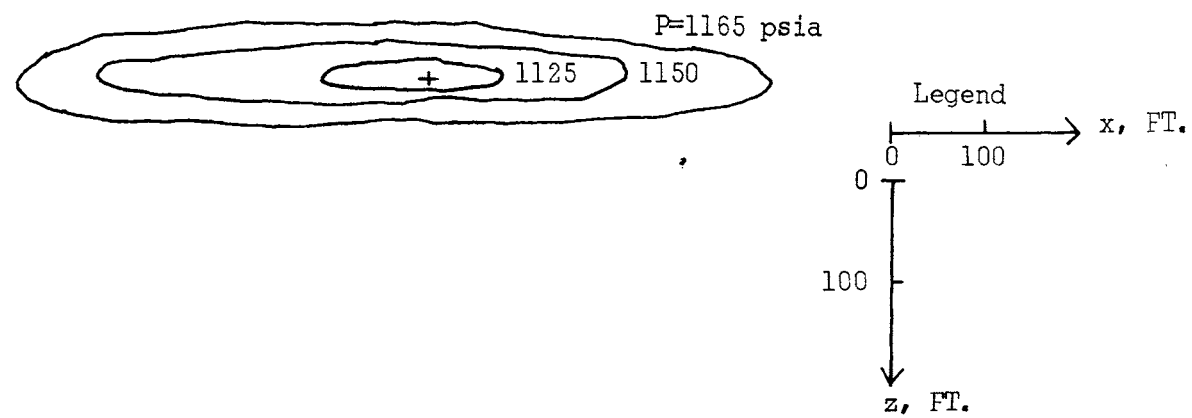


Figure 9 Pressure Distribution Surrounding a Producing Well in a
Bottom-Water Drive Reservoir After 1188 Days, x-z Plane

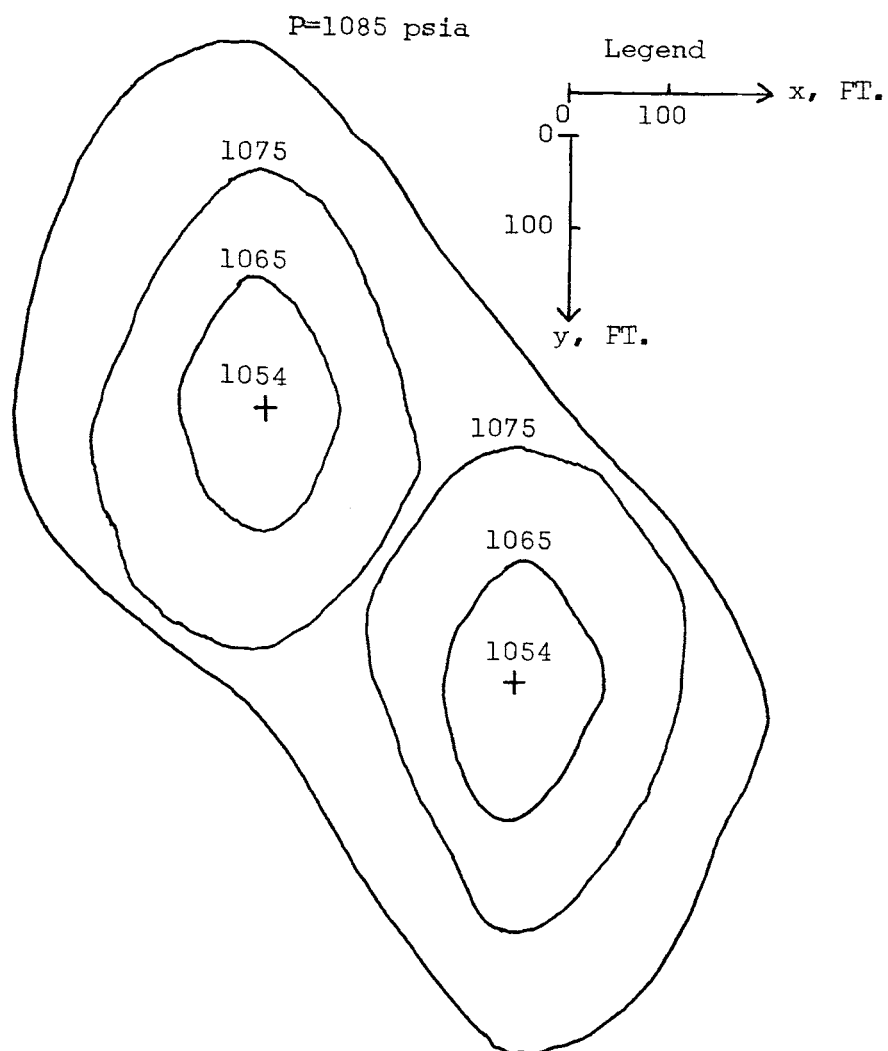


Figure 10 Pressure Distribution Surrounding Two Producing Wells
in a Bottom-Water Drive Reservoir After 478 Days, x-y
Plane

V. CONCLUSIONS

The model developed herein describes in detail three phase flow phenomena (pressure and saturation distributions) in three dimensions for total reservoirs and individual wells. A rapid and stable solution is shown for the non-linear partial differential equations which describe these conditions. Although the model was developed in terms of rectangular coordinates, the radial flow which exists in the vicinity of wellbores is adequately described. Fields employing water and/or gas injection, bottom-water drive, aquifer drive, gas-cap expansion drive, solution gas drive or combinations of the above are realistically simulated by the model. Each of these cases has been studied with favorable results.

The P. L. T. Brian technique¹ was successfully applied to the solution of the non-linear equations which arise in reservoir modelling by superimposing an implicit, iterative scheme on the technique for calculating coefficients. Brian's suggested adaptations for solving non-linear equations were found to be inadequate for solving the pressure simulation equations. His suggestions were based on his findings for heat conduction problems which contain no function as complex as the relative permeability equations encountered in fluid flow. This accounts for the fact that it was necessary to devise new techniques for the fluid flow problem.

It was found that oil and water relative permeabilities based

on upstream saturations and gas relative permeabilities based on interblock average saturations yield the most realistic results. This conclusion was arrived at by observation of modelling results. It has not been found in the literature, but discussions with modelling specialists in the petroleum industry have led to the conclusion that most use this scheme in two dimensional studies. These independent observations tend to validate the use of the technique in this model, and the use here expands the feasibility of the method to a three dimensional model.

It was concluded that a stable three dimensional model applicable to general usage is feasible, as illustrated by the model developed for this study.

VI. APPENDICES

APPENDIX A

Potential Determinations

Defining P as the pressure in the oil phase and following the convention shown in Figure 2 (z is positive downward), the total potential for the oil phase is written as

$$\Phi_o = P - (p_o \Delta z / 144) . \quad (A-1)$$

Similarly, assuming no oil-gas capillarity, the resulting gas potential is

$$\Phi_g = P - (p_g \Delta z / 144) . \quad (A-2)$$

And, for the water phase,

$$P_w = P - P_c . \quad (A-3)$$

This yields an equation for the water potential as follows:

$$\Phi_w = P - P_c - (p_w \Delta z / 144) . \quad (A-4)$$

APPENDIX B

Approximation of the Pressure Equation

Application of a forward finite difference approximation to Equation (12) as well as various mathematical manipulations yields, in the x-direction:

$$\begin{aligned} & \left[AX_i P_{i-1}^{n+\frac{1}{2}} + BX_i P_i^{n+\frac{1}{2}} + CX_i P_{i+1}^{n+\frac{1}{2}} \right]_{jk} \\ & + \left[AY_j P_{j-1}^n + BY_j P_j^n + CY_j P_{j+1}^n \right]_{ik} \\ & + \left[AZ_k P_{k-1}^n + BZ_k P_k^n + CZ_k P_{k+1}^n \right]_{ij} = D_{ijk} , \quad (B-1) \end{aligned}$$

where the coefficients are defined as follows:

$$AX_{ijk} = 6.3225 \left[\frac{M_o}{p_o} \frac{i-\frac{1}{2}}{i} + \frac{M_w}{p_w} \frac{i-\frac{1}{2}}{i} + \frac{M_g}{p_g} \frac{i-\frac{1}{2}}{i} \right]_{jk} ; \quad (B-2)$$

$$CX_{ijk} = 6.3225 \left[\frac{M_o}{p_o} \frac{i+\frac{1}{2}}{i} + \frac{M_w}{p_w} \frac{i+\frac{1}{2}}{i} + \frac{M_g}{p_g} \frac{i+\frac{1}{2}}{i} \right]_{jk} ; \quad (B-3)$$

$$BX_{ijk} = -AX_{ijk} - CX_{ijk} - \frac{2 \Delta x_i \Delta y_j \Delta z_k \phi_{ijk}}{\Delta t} G_{ijk} . \quad (B-4)$$

A point term for compressibility and solution gas is needed to facilitate the definitions of the B and D coefficients. This term is arbitrarily called G and is defined as follows:

$$G_{ijk} = (c_r + S_w c_w + S_o c_o + \frac{S_g}{p_g} \frac{\partial p_g}{\partial P})_{ijk} , \text{ for } P > P_{bp} . \quad (B-5a)$$

$$G_{ijk} = (c_r + S_w c_w + \frac{S_o}{p_o} \frac{\partial p_o}{\partial P} + \frac{S_g}{p_g} \frac{\partial p_g}{\partial P} + \frac{S_o B_g}{5.615 B_o} \frac{\partial R_s}{\partial P})_{ijk} , \text{ for } P \leq P_{bp} . \quad (B-5b)$$

AY, CY, AZ and CZ are defined in the same manner as AX and CX.

$$B_{Y_{ijk}} = -A_{Y_{ijk}} - C_{Y_{ijk}} + \frac{2 \Delta x_i \Delta y_j \Delta z_k \phi_{ijk}}{\Delta t} G_{ijk} ; \quad (B-6)$$

$$B_{Z_{ijk}} = -A_{Z_{ijk}} - C_{Z_{ijk}} ; \quad (B-7)$$

$$D_{ijk} = PCX_{ijk} + PCY_{ijk} + PCZ_{ijk} + QTERM_{ijk} + GZ_{ijk} ; \quad (B-8)$$

$$PCX_{ijk} = (6.3225/p_w)_{ijk} \left[M_{w \ i-\frac{1}{2}} P_{c \ i-1} - (M_{w \ i-\frac{1}{2}} + M_{w \ i+\frac{1}{2}}) P_{c \ i} + M_{w \ i+\frac{1}{2}} P_{c \ i+1} \right]_{jk} . \quad (B-9)$$

PCY and PCZ are similarly designated.

$$QTERM_{ijk} = 5.615(q_o B_o + q_w B_w)_{ijk} + (Q_g B_g)_{ijk} ; \quad (B-10)$$

$$GZ_{ijk} = (GZ_o + GZ_w + GZ_g)_{ijk} ; \quad (B-11)$$

$$GZ_o)_{ijk} = \frac{6.3225}{144 p_o)_{ijk}} \left[(M_o p_o)_{k+\frac{1}{2}} \frac{1}{2} (\Delta z_{k+1} + \Delta z_k) - (M_o p_o)_{k-\frac{1}{2}} \frac{1}{2} (\Delta z_k + \Delta z_{k-1}) \right]_{ij} . \quad (B-12)$$

The water and gas phase terms are analogously denoted. Concerning the oil phase in Equation (B-2), $M_o)_{i-\frac{1}{2},jk}$ is defined as

$$M_o)_{i-\frac{1}{2},jk} = \frac{2 \Delta y_j \Delta z_k k_{i-\frac{1}{2}}}{\Delta x_i + \Delta x_{i-1}} \left[\frac{k r_o p_o}{u_o} \right]_{i-\frac{1}{2},jk}^{n+\frac{1}{2}} \quad (B-13)$$

The M-terms of the remaining oil, water and gaseous phases in the x-, y- and z-directions are of the same form as Equation (B-13).

Corresponding equations are then solved in the y- and z-directions, respectively, for the ultimate pressure solution.

A simple linear extrapolation based on time increments is used to determine the predicted pressures.

APPENDIX C

Material Balance Solution

Each phase saturation may be represented on a per block basis by the relationship,

$$S_{ijk}^{n+1} = S_{ijk}^n \left[\frac{\phi_{pn}^{n+1}}{\phi_{p^{n+1}}^{n+1}} \right]_{ijk} + \Delta S_{ijk} \quad (C-1)$$

where the general equation for the water and oil phase saturation changes is of the following type:

$$\Delta S_{ijk} = \frac{\text{mass in} - \text{mass out} - \text{mass produced}}{\Delta x_i \Delta y_j \Delta z_k \phi_{ijk}^{n+1} p_{ijk}^{n+1}} \quad (C-2)$$

The change in saturation of the water phase may be expressed as

$$\begin{aligned} \Delta S_{w\ ijk} = & \left[\frac{\Delta t}{\Delta x_i \Delta y_j \Delta z_k (\phi_{pw})_{ijk}^{n+1}} \right] \left[-6.3225 \right. \\ & (N_{w\ i-\frac{1}{2},jk} + N_{w\ i,j-\frac{1}{2},k} + N_{w\ ij,k-\frac{1}{2}} - N_{w\ i+\frac{1}{2},jk} \\ & - N_{w\ i,j+\frac{1}{2},k} - N_{w\ ij,k+\frac{1}{2}}) \\ & \left. - 5.615(q_w B_w p_w)_{ijk}^{n+\frac{1}{2}} \right] \quad (C-3) \end{aligned}$$

and the arbitrarily selected coefficient N is defined as follows:

$$N_{w\ i-\frac{1}{2},jk} = M_{w\ i-\frac{1}{2},jk} (P_i - P_{i-1} - P_{ci} + P_{ci-1})_{jk}^{n+\frac{1}{2}} \quad (C-4)$$

where M has been previously designated in Appendix B. The oil phase saturation change equation is similar to that for water with the exception that no capillary pressures appear and all properties

apply to oil.

The general equation for the gas phase varies slightly from the other phases:

$$\Delta S_{g\ ijk} = \frac{1}{\Delta x_i \Delta y_j \Delta z_k} \phi_{ijk}^{n+1} (\text{mass in} - \text{mass out} - \text{mass produced} + \text{mass evolved}) / p_{ijk}^{n+1} \quad (C-5)$$

The equation for the gas saturation incrementation may be written as:

$$\Delta S_{g\ ijk} = \frac{\Delta t}{\Delta x_i \Delta y_j \Delta z_k} (\phi_{pg})_{ijk}^{n+1} \left[-6.3225 \right. \\ \left. (N_{g\ i-\frac{1}{2},jk} + N_{g\ i,j-\frac{1}{2},k} + N_{g\ ij,k-\frac{1}{2}} - N_{g\ i+\frac{1}{2},jk} \right. \\ \left. - N_{g\ i,j+\frac{1}{2},k} - N_{g\ ij,k+\frac{1}{2}}) \right. \\ \left. - (Q_{gB_g p_g})_{ijk}^{n+\frac{1}{2}} \right] - \left(\frac{\Delta R s S_o B_g p_g}{5.615 B_o} \right)_{ijk}^{n+\frac{1}{2}} \left(\frac{1}{p_g} \right)_{ijk}^{n+1} \quad (C-6)$$

Once again, as with the oil phase, no capillary pressures appear in the gas phase, yielding an N-term identical (except for phase notation) to the oil phase:

$$N_{g\ i-\frac{1}{2},jk} = M_{g\ i-\frac{1}{2},jk} (P_i - P_{i-1})_{jk}^{n+\frac{1}{2}} \quad (C-7)$$

Due to the fact that all pressure calculations are completed prior to saturation determinations, all pressures are known and an explicit solution is applicable to the saturations.

APPENDIX D

Physical Property Equations

1. Relative Permeabilities

The relative permeability equations for the water and oil phases are those employed by Gottfried.¹³

$$kr_g = \left[\frac{S_g - S_{gc}}{1 - S_{gc}} \right]^4 ; \quad (D-1)$$

$$kr_o = S_o^3 (2 - S_o - 2S_{wc}) / (1 - S_{wc})^4 ; \quad (D-2)$$

$$kr_w = \left[\frac{S_w - S_{wc}}{1 - S_{wc}} \right]^4 , \text{ for } S_w > S_{wc} ; \quad (D-3)$$

$$kr_w = 0 , \text{ for } S_w \leq S_{wc} . \quad (D-4)$$

$$kr^{n+1} = kr^n + \frac{\partial kr}{\partial S} (S^{n+1} - S^n) . \quad (D-5)$$

2. Densities

Densities of gas, water and oil at or below the bubble point pressure are calculated internally from the following type of polynomial equation:

$$p = a_0 + a_1 P + \dots + a_4 P^4 . \quad (D-6)$$

Oil densities at pressures above the bubble point are determined from the relationship:

$$p_o = (a_0 + a_1 P_{bp} + \dots + a_4 P_{bp}^4) e^{c_o(P - P_{bp})} . \quad (D-7)$$

3. Viscosities

Viscosities of gas, water and oil at or below the bubble point pressure are calculated from an equation of the ensuing sort:

$$u = a_0 + a_1 P + \dots + a_4 P^4 . \quad (D-8)$$

Oil viscosities at pressures above the bubble point are ascertained from the relationship

$$u_o = (a_0 + a_1 P_{bp} + \dots + a_4 P_{bp}^4) + \frac{\partial u}{\partial P} (\Delta P) . \quad (D-9)$$

4. Solution Gas

$$R_s = a_0 + a_1 P + \dots + a_4 P^4 , \text{ for } P \leq P_{bp} . \quad (D-10)$$

5. Formation Volume Factors

$$B_g = 1 / (a_0 + a_1 P + \dots + a_4 P^4) ; \quad (D-11)$$

$$B_w = (a_0 + a_1 P + \dots + a_4 P^4) e^{-c_w(P - P_{bp})} ; \quad (D-12)$$

$$B_o = a_0 + a_1 P + \dots + a_4 P^4 , \text{ for } P \leq P_{bp}; \quad (D-13)$$

$$B_o = (a_0 + a_1 P_{bp} + \dots + a_4 P_{bp}^4) e^{-c_o(P - P_{bp})} , \quad (D-14)$$

for $P > P_{bp} .$

6. Capillary Pressure

$$P_c = P_{c_{max}} , \text{ for } S_w \leq S_{wc} ; \quad (D-15)$$

$$P_c = (a_0 + a_1 S_w)^{-\frac{1}{2}} , \text{ for } S_{wc} < S_w \leq S_{w_{max}} ; \quad (D-16)$$

$$P_c = (a_0 + a_1 S_w)^{-\frac{1}{2}} (1 - S_w) / (1 - S_{w \max}) ,$$

$$\text{for } 1 \geq S_w > S_{w \max} . \quad (D-17)$$

7. Porosity

$$\phi^{n+1} = \phi^n e^{c_r (P^{n+1} - P^n)} . \quad (D-18)$$

The term c_r represents the pore volume compressibility and has units of pore volume/pore volume/psi where the pore volumes are in cubic feet.

APPENDIX E

Nomenclature

Capital Letters

A	=	cross-sectional area, sq. ft.
B	=	formation volume factor, res. vol./std. vol.
P	=	pressure, psia.
P _{bp}	=	bubble point pressure, psia.
P _c	=	capillary pressure, psi.
Q	=	flow rate, cu. ft./day, (std. conditions).
R _s	=	solution gas-oil ratio, std. cu. ft./ std. bbl.
S	=	saturation, fraction.

Lowercase Letters

a	=	fluid properties constant, various units.
c	=	compressibility, 1/psi.
k	=	absolute permeability, darcys.
k _r	=	relative permeability, fraction.
p	=	density, lb./res. cu. ft.
q	=	flow rate, std. bbl./day.
t	=	time, days.
u	=	viscosity, cp.
x	=	length, feet.
y	=	width, feet.
z	=	depth, feet.

Greek Symbols

Δ	=	increment.
Φ	=	potential, psi.
ϕ	=	porosity, fraction.

Subscripts

i, j, k	=	individual block directional parameters (see Figure 2).
g	=	gas.
gc	=	irreducible gas.
o	=	oil.
or	=	residual oil.
r	=	rock.
w	=	water.
wc	=	connate water.
x, y, z	=	total directional parameters (see Figure 1), feet.

Superscripts

n	=	current time level.
$*$	=	pseudo-pressure, psia.

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